10.4 SECANTS AND TANGENTS

Objectives

After studying this section, you will be able to

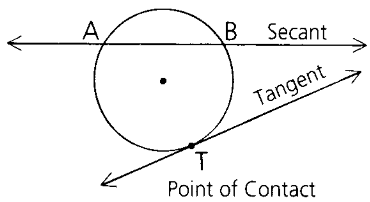
- Identify secant and tangent lines
- Identify secant and tangent segments
- Distinguish between two types of tangent circles
- Recognize common internal and common external tangents

Part One: Introduction

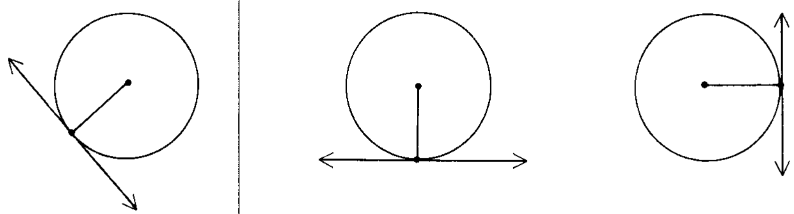
Secant and Tangent Lines

Some lines and circles have special relationships.

Definition A ***secant*** is a line that intersects a circle at exactly two points. (Every secant contains a chord of the circle.)



Definition A ***tangent*** is a line that intersects a circle at exactly one point. This point is called the ***point of tangency*** or ***point of contact***.



The diagrams above suggest the following postulates about tangents.

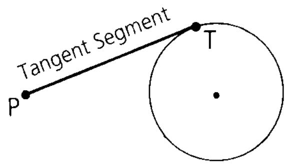
Postulate ***A tangent line is perpendicular to the radius drawn to the point of contact.***

Postulate ***If a line is perpendicular to a radius at its outer endpoint, then it is tangent to the circle.***

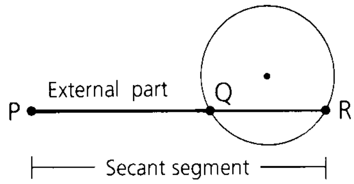
Secant and Tangent Segments

Some segments are related to circles in similar ways.

Definition A ***tangent segment*** is the part of a tangent line between the point of contact and a point outside the circle.



Definition A ***secant segment*** is the part of a secant line that joins a point outside the circle to the farther intersection point of the secant and the circle.

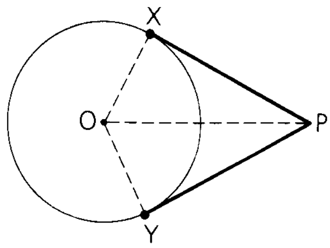


Definition The ***external part*** of a secant segment is the part of a secant line that joins the outside point to the nearer intersection point.

Theorem 85 ***If two tangent segments are drawn to a circle from an external point, then those segments are congruent. (Two-Tangent Theorem)***

Given: ⊙O;
$\overline{PX}$ and $\overline{PY}$ are tangent segments.

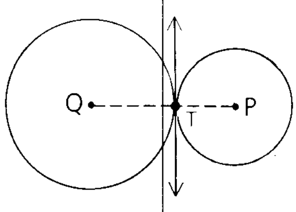
Prove: $\overline{PX} \cong \overline{PY}$

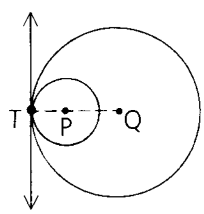


The Two-Tangent Theorem is easily proved with congruent triangles. More theorems relating to secant segments and tangent segments are presented in Section 10.8.

Tangent Circles

Definition ***Tangent circles*** are circles that intersect each other at exactly one point.





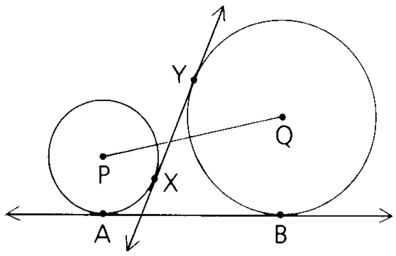
Definition Two circles are ***externally tangent*** if each of the tangent circles lies outside the other. (See the left-hand figure above.)

Definition Two circles are ***internally tangent*** if one of the tangent circles lies inside the other. (See the right-hand figure on the preceding page.)

Notice that in each case the tangent circles have one common tangent at their point of contact. Also, the point of contact lies on the ***line of centers,*** $\overleftrightarrow{PQ}$.

Common Tangents

$\overleftrightarrow{PQ}$ is the line of centers.
$\overleftrightarrow{XY}$ is a ***common internal tangent.***
$\overleftrightarrow{AB}$ is a ***common external tangent.***

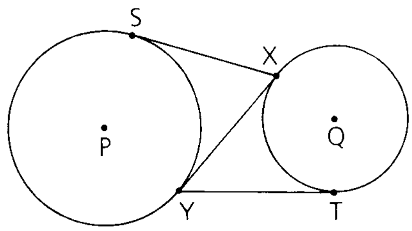


Definition A ***common tangent*** is a line tangent to two circles (not necessarily at the same point). Such a tangent is a ***common internal tangent*** if it lies between the circles (intersects the segment joining the centers) or a ***common external tangent*** if it is not between the circles (does not intersect the segment joining the centers).

In practice, we will frequently refer to a *segment* as a common tangent if it lies on a common tangent and its endpoints are the tangent's points of contact. In the preceding diagram, for example, $\overline{XY}$ can be called a common internal tangent and $\overline{AB}$ can be called a common external tangent.

Part Two: Sample Problems

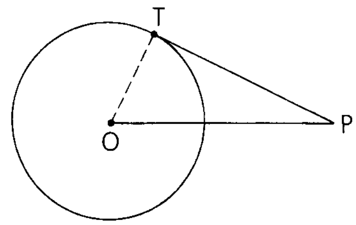
Problem 1 *Given:* $\overline{XY}$ *is a common internal tangent to* ⓢ P *and* Q *at* X *and* Y.
$\overline{XS}$ *is tangent to* ⊙P *at* S.
$\overline{YT}$ *is tangent to* ⊙Q *at* T.
Conclusion: $\overline{XS} \cong \overline{YT}$



Proof

1 $\overline{XS}$ is tangent to ⊙P. $\overline{YT}$ is tangent to ⊙Q.	1 Given
2 $\overline{XY}$ is tangent to ⓢ P and Q.	2 Given
3 $\overline{XS} \cong \overline{XY}$	3 Two-Tangent Theorem
4 $\overline{XY} \cong \overline{YT}$	4 Same as 3
5 $\overline{XS} \cong \overline{YT}$	5 Transitive Property

Problem 2 *$\overleftrightarrow{TP}$ is tangent to circle O at T. The radius of circle O is 8 mm. Tangent segment $\overline{TP}$ is 6 mm long. Find the length of $\overline{OP}$.*

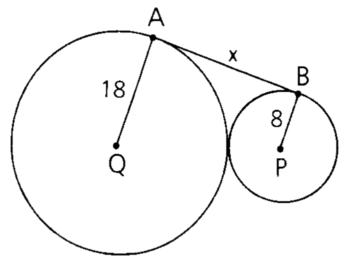


Solution Draw radius $\overline{OT}$ to form right triangle OTP.

$$(TP)^2 + (TO)^2 = (OP)^2$$
$$6^2 + 8^2 = (OP)^2$$
$$\pm 10 = OP \qquad \text{(Reject } -10.)$$

Thus, OP = 10 mm.

Problem 3 *A circle with a radius of 8 cm is externally tangent to a circle with a radius of 18 cm. Find the length of a common external tangent.*



Solution There is a standard procedure for solving a problem involving a common tangent (either internal or external).

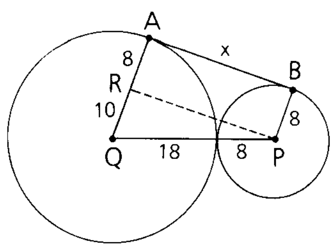
Common-Tangent Procedure

1. Draw the segment joining the centers.
2. Draw the radii to the points of contact.
3. Through the center of the smaller circle, draw a line parallel to the common tangent.
4. Observe that this line will intersect the radius of the larger circle (extended if necessary) to form a rectangle and a right triangle.
5. Use the Pythagorean Theorem and properties of a rectangle.

In $\triangle RPQ$,

$$(QR)^2 + (RP)^2 = (PQ)^2$$
$$10^2 + (RP)^2 = 26^2$$
$$RP = \pm 24$$

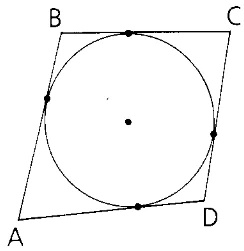
Thus, AB = 24 cm.



Problem 4 *A walk-around problem:*

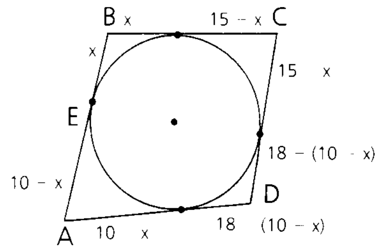
Given: *Each side of quadrilateral ABCD is tangent to the circle.*
AB = 10, BC = 15, AD = 18

Find: CD



Solution Let BE = x and "walk around" the figure, using the given information and the Two-Tangent Theorem.

$$\begin{aligned} CD &= 15 - x + 18 - (10 - x) \\ &= 15 - x + 18 - 10 + x \\ &= 23 \end{aligned}$$

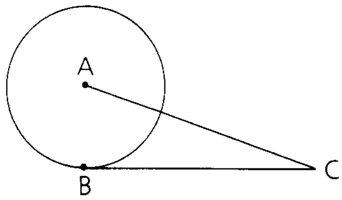


See problems 16, 21, 22, and 29 for other types of walk-around problems.

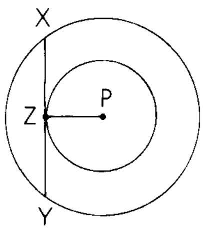
Part Three: Problem Sets

Problem Set A

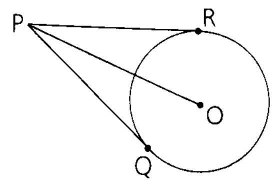
1 The radius of ⊙A is 8 cm.
Tangent segment $\overline{BC}$ is 15 cm long.
Find the length of $\overline{AC}$.



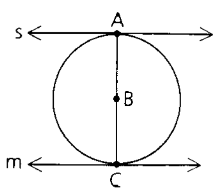
2 Concentric circles with radii 8 and 10 have center P.
$\overline{XY}$ is a tangent to the inner circle and is a chord of the outer circle.
Find $\overline{XY}$. (Hint: Draw $\overline{PX}$ and $\overline{PY}$.)



3 Given: $\overline{PR}$ and $\overline{PQ}$ are tangents to ⊙O at R and Q.
Prove: $\overrightarrow{PO}$ bisects ∠RPQ. (Hint: Draw $\overline{RO}$ and $\overline{OQ}$.)



4 Given: $\overline{AC}$ is a diameter of ⊙B.
Lines s and m are tangents to the ⊙ at A and C.
Conclusion: s ∥ m

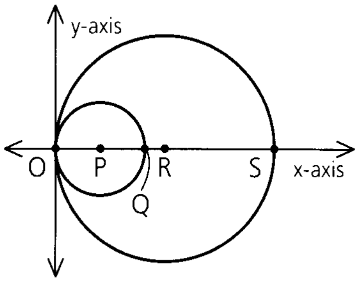


Problem Set A, *continued*

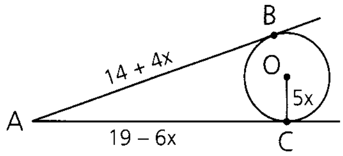
5 ⊙P and ⊙R are internally tangent at O. P is at (8, 0) and R is at (19, 0).

a Find the coordinates of Q and S.

b Find the length of $\overline{QR}$.



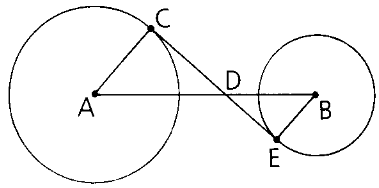
6 $\overline{AB}$ and $\overline{AC}$ are tangents to ⊙O, and OC = 5x. Find OC.



7 Given: $\overline{CE}$ is a common internal tangent to circles A and B at C and E.

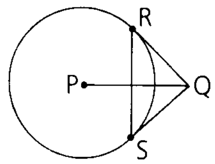
Prove: **a** $\angle A \cong \angle B$

b $\frac{AD}{BD} = \frac{CD}{DE}$



8 Given: $\overline{QR}$ and $\overline{QS}$ are tangent to ⊙P at points R and S.

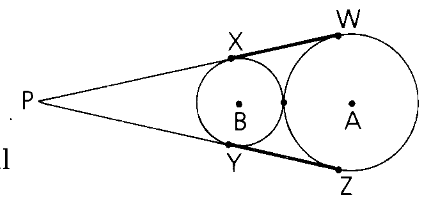
Prove: $\overline{PQ} \perp \overline{RS}$ (Hint: This can be proved in just a few steps.)



9 Given: $\overline{PW}$ and $\overline{PZ}$ are common tangents to ⓢ A and B at W, X, Y, and Z.

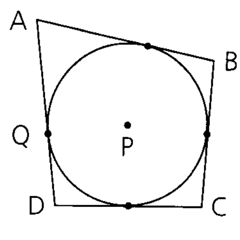
Prove: $\overline{WX} \cong \overline{YZ}$ (Hint: No auxiliary lines are needed.)

Note This is part of the proof of a useful property: The common external tangent segments of two circles are congruent.



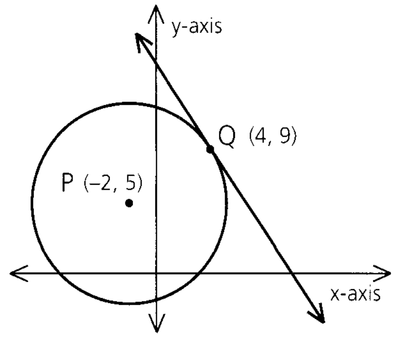
Problem Set B

10 ⊙P is tangent to each side of ABCD. AB = 20, BC = 11, and DC = 14. Let AQ = x and find AD.



11 **a** Find the radius of $\odot$P.

b Find the slope of the tangent to $\odot$P at point Q.



12 Two concentric circles have radii 3 and 7. Find, to the nearest hundredth, the length of a chord of the larger circle that is tangent to the smaller circle. (See problem 2 for a diagram.)

13 The centers of two circles of radii 10 cm and 5 cm are 13 cm apart.

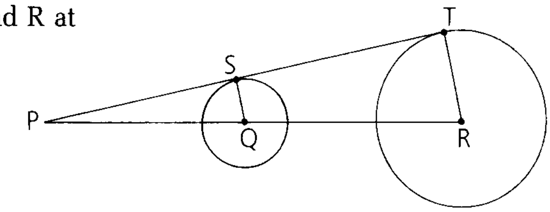
a Find the length of a common external tangent. (Hint: Use the common-tangent procedure.)

b Do the circles intersect?

14 The centers of two circles with radii 3 and 5 are 10 units apart. Find the length of a common internal tangent. (Hint: Use the common-tangent procedure.)

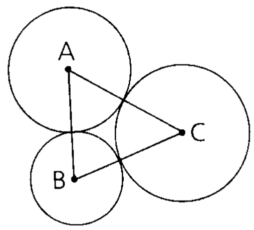
15 Given: $\overline{PT}$ is tangent to ⓢ Q and R at points S and T.

Conclusion: $\frac{PQ}{PR} = \frac{SQ}{TR}$



16 Given: Tangent ⓢ A, B, and C, AB = 8, BC = 13, AC = 11

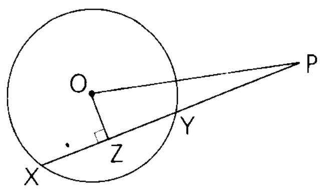
Find: The radii of the three ⓢ (Hint: This is a walk-around problem.)



17 The radius of $\odot$O is 10. The secant segment $\overline{PX}$ measures 21 and is 8 units from the center of the $\odot$.

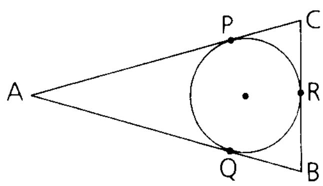
a Find the external part (PY) of the secant segment.

b Find OP.



Problem Set B, *continued*

18 Given: $\triangle ABC$ is isosceles, with base $\overline{BC}$.
Conclusion: $\overline{BR} \cong \overline{RC}$

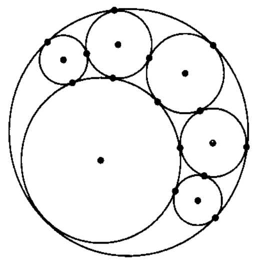


19 If two of the seven circles are chosen at random, what is the probability that the chosen pair are

a Internally tangent?

b Externally tangent?

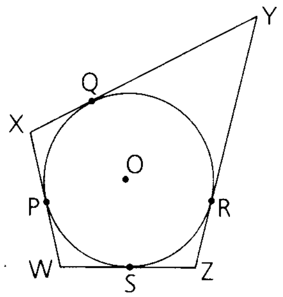
c Not tangent?



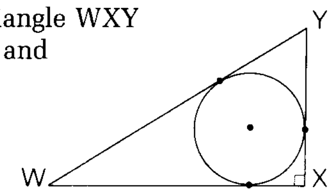
20 Find, to the nearest tenth, the distance between two circles if their radii are 1 and 4 and the length of a common external tangent is $7\frac{1}{2}$.

Problem Set C

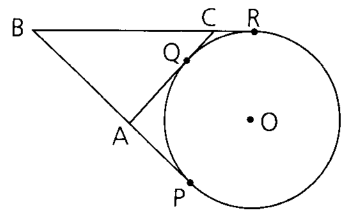
21 Given: Quadrilateral WXYZ is *circumscribed about* $\odot$O (that is, its sides are tangent to the $\odot$).
Prove: XY + WZ = WX + YZ



22 Find the perimeter of right triangle WXY if the radius of the circle is 4 and WY = 20.



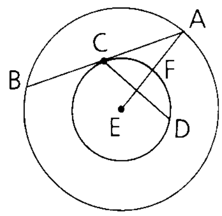
23 B is 34 mm from the center of circle O, which has radius 16 mm. $\overline{BP}$ and $\overline{BR}$ are tangent segments. $\overleftrightarrow{AC}$ is tangent to $\odot$O at point Q. Find the perimeter of $\triangle ABC$.



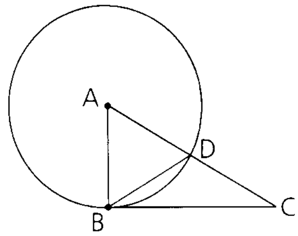
24 Find the coordinates of the center of a circle that is tangent to the y-axis and intersects the x-axis at (8, 0) and (18, 0).

25 Given: Two concentric circles with center E, $AB = 40$, $CD = 24$, $\overline{CD} \perp \overline{AE}$; $\overline{AB}$ is tangent at C.

Find: AF

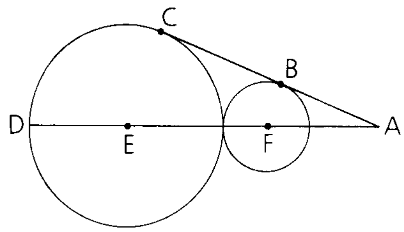


26 $\overline{BC}$ is tangent to $\odot A$ at B, and $\overline{BD} \cong \overline{BA}$. Explain why $\overline{BD}$ bisects $\overline{AC}$.

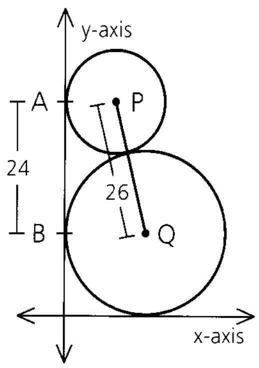


27 Given: ⓢ E and F, with $\overline{AC}$ tangent at B and C, $DE = 10$, $FB = 4$

Find: AB

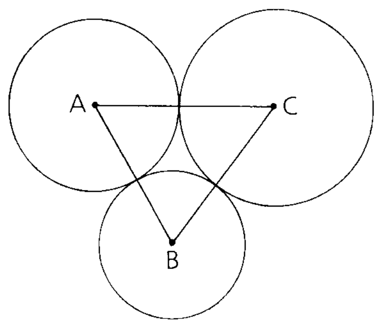


28 Circles P and Q are tangent to each other and to the axes as shown. $PQ = 26$ and $AB = 24$. Find the coordinates of P and Q.



29 Given: Three tangent ⓢ, A, B, and C, $BC = a$, $AC = b$, $AB = c$

Find: The radius of $\odot A$ in terms of a, b, and c



10.5 ANGLES RELATED TO A CIRCLE

Objectives

After studying this section, you will be able to

- Determine the measures of central angles
- Determine the measures of inscribed and tangent-chord angles
- Determine the measures of chord-chord angles
- Determine the measures of secant-secant, secant-tangent, and tangent-tangent angles

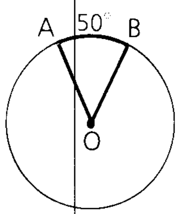
Part One: Introduction

Angles with Vertices at the Center of a Circle

The measure of an angle whose sides intersect a circle is determined by the measure of its intercepted arcs. The location of the vertex of each angle is the key to remembering how to compute the measure of the angle.

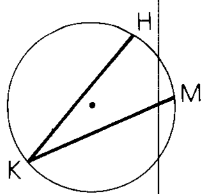
An angle with its vertex at the center of a circle is a central angle, already defined to be equal in measure to its intercepted arc (Section 10.3).

In ⊙O, $\overset{\frown}{AB} = 50°$, so $m\angle AOB = 50$.

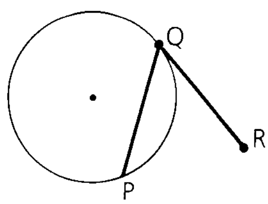


Angles with Vertices on a Circle

Two important types of angles whose vertices are on a circle are shown below.



$\angle HKM$ is an ***inscribed angle.***



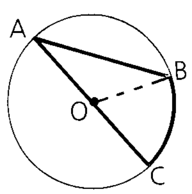
$\angle PQR$ is a ***tangent-chord angle.***

Definition An ***inscribed angle*** is an angle whose vertex is on a circle and whose sides are determined by two chords.

Definition A ***tangent-chord angle*** is an angle whose vertex is on a circle and whose sides are determined by a tangent and a chord that intersect at the tangent's point of contact.

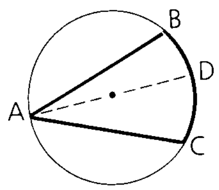
Theorem 86 ***The measure of an inscribed angle or a tangent-chord angle (vertex on a circle) is one-half the measure of its intercepted arc.***

The proof of Theorem 86 for inscribed angles is unusual because three cases must be considered. Shown below are some key steps for each case in the proof that $m\angle BAC = \frac{1}{2}(m\overset{\frown}{BC})$.



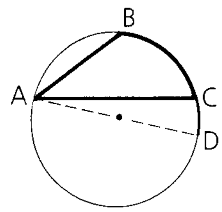
Case 1:
The center lies on a side of the angle.

1 $m\angle BOC = m\overset{\frown}{BC}$
2 $\angle BOC = \angle BAC + \angle ABO$, so $m\angle BOC = 2(m\angle BAC)$



Case 2:
The center lies inside the angle.

1 Use case 1 twice.
2 Add $\angle$s and arcs.

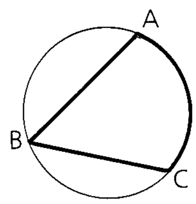


Case 3:
The center lies outside the angle.

1 Use case 1 twice.
2 Subtract $\angle$s and arcs.

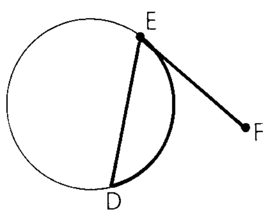
Example 1 *Given:* $m\overset{\frown}{AC} = 112$
Find: $m\angle B$

$$m\angle B = \frac{1}{2}(m\overset{\frown}{AC}) = \frac{1}{2} \cdot 112 = 56$$



Example 2 *Given:* $\overline{FE}$ *is tangent at* E.
$m\overset{\frown}{DE} = 80$
Find: $m\angle DEF$

$$m\angle DEF = \frac{1}{2}(m\overset{\frown}{DE}) = \frac{1}{2} \cdot 80 = 40$$

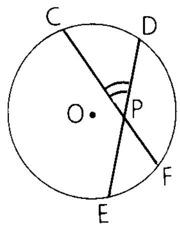


Angles with Vertices Inside, but Not at the Center of, a Circle

One type of angle other than a central angle has a vertex inside a circle.

Definition A ***chord-chord angle*** is an angle formed by two chords that intersect inside a circle but not at the center.

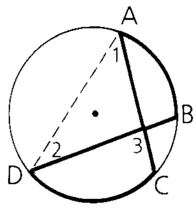
$\angle CPD$ is one of four chord-chord angles formed by chords $\overline{CF}$ and $\overline{DE}$ in circle O.



Theorem 87 ***The measure of a chord-chord angle is one-half the sum of the measures of the arcs intercepted by the chord-chord angle and its vertical angle.***

Notice that one-half the sum of the arc measures is the same as the *average* of the arc measures.

Given: $\angle 3$ is a chord-chord angle.
Prove: $m\angle 3 = \frac{1}{2}(m\overset{\frown}{AB} + m\overset{\frown}{CD})$



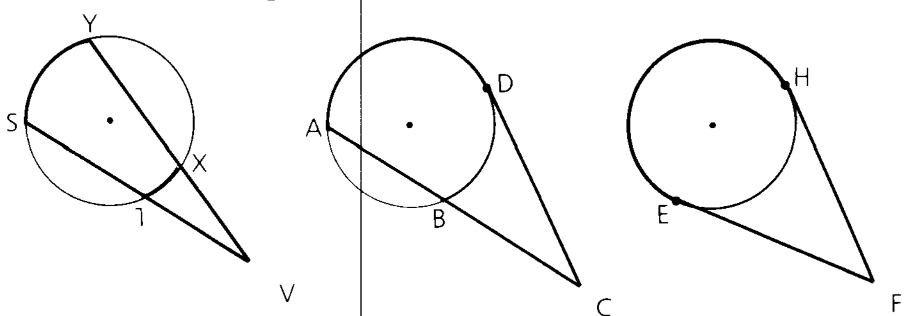
Here are two key steps in a proof of Theorem 87.

1 $m\angle 3 = m\angle 1 + m\angle 2$

2 $m\angle 3 = \frac{1}{2}(m\overset{\frown}{CD}) + \frac{1}{2}(m\overset{\frown}{AB})$

Angles with Vertices Outside a Circle

There are three types of angles having a vertex outside a circle and both sides intersecting the circle.



$\angle V$ is a ***secant-secant angle.*** $\angle C$ is a ***secant-tangent angle.*** $\angle F$ is a ***tangent-tangent angle.***

Definition A ***secant-secant angle*** is an angle whose vertex is outside a circle and whose sides are determined by two secants.

Definition A ***secant-tangent angle*** is an angle whose vertex is outside a circle and whose sides are determined by a secant and a tangent.

Definition A ***tangent-tangent angle*** is an angle whose vertex is outside a circle and whose sides are determined by two tangents.

Theorem 88 ***The measure of a secant-secant angle, a secant-tangent angle, or a tangent-tangent angle (vertex outside a circle) is one-half the difference of the measures of the intercepted arcs.***

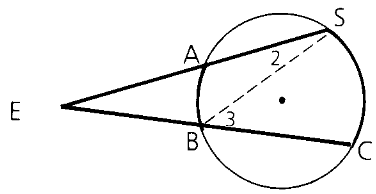
Key steps in a proof of Theorem 88 for secant-secant angles follow.

Prove: $m\angle E = \frac{1}{2}(m\widehat{SC} - m\widehat{AB})$

1 $m\angle 3 = m\angle E + m\angle 2$; solve for $m\angle E$.

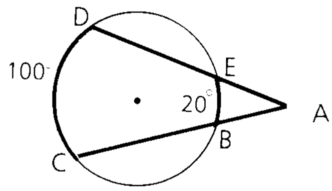
2 $m\angle 2 = \frac{1}{2}(m\widehat{AB})$; $m\angle 3 = \frac{1}{2}(m\widehat{SC})$

3 Substitute and simplify.



Example 1 *Find* $m\angle A$.

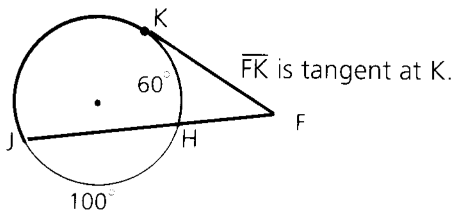
$$m\angle A = \tfrac{1}{2}(m\widehat{CD} - m\widehat{BE}) = \tfrac{1}{2}(100 - 20) = 40$$



Example 2 *Find* $m\angle F$.

$$m\widehat{JK} = 360 - 100 - 60 = 200$$

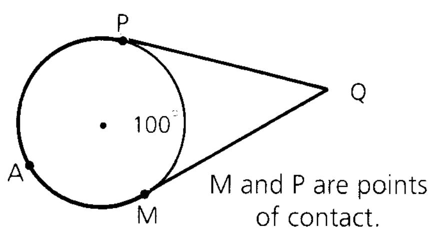
$$m\angle F = \tfrac{1}{2}(m\widehat{JK} - m\widehat{HK}) = \tfrac{1}{2}(200 - 60) = 70$$



Example 3 *Find* $m\angle Q$.

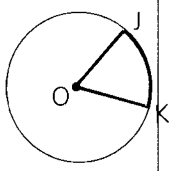
$$m\widehat{MAP} = 360 - 100 = 260$$

$$m\angle Q = \tfrac{1}{2}(m\widehat{MAP} - m\widehat{MP}) = \tfrac{1}{2}(260 - 100) = 80$$



Angle-Arc Summary

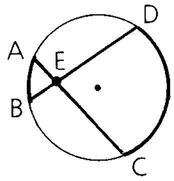
Central Angle



$m\angle KOJ = m\overset{\frown}{JK}$

Vertex at center $\Rightarrow$ equal

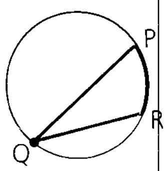
Chord-Chord Angle



$m\angle DEC = \frac{1}{2}(m\overset{\frown}{AB} + m\overset{\frown}{CD})$

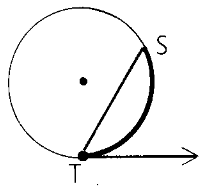
Vertex inside $\Rightarrow$ half the sum

Inscribed Angle



$m\angle Q = \frac{1}{2}(m\overset{\frown}{PR})$

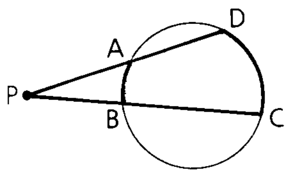
Tangent-Chord Angle



$m\angle T = \frac{1}{2}(m\overset{\frown}{ST})$

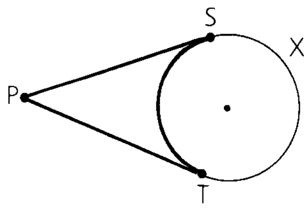
Vertex on circle $\Rightarrow$ half the arc

Secant-Secant Angle



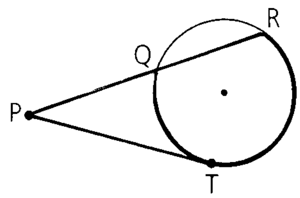
$m\angle P = \frac{1}{2}(m\overset{\frown}{CD} - m\overset{\frown}{AB})$

Tangent-Tangent Angle



$m\angle P = \frac{1}{2}(m\overset{\frown}{SXT} - m\overset{\frown}{ST})$

Secant-Tangent Angle

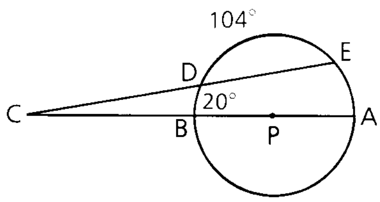


$m\angle P = \frac{1}{2}(m\overset{\frown}{RT} - m\overset{\frown}{QT})$

Vertex outside circle $\Rightarrow$ half the difference

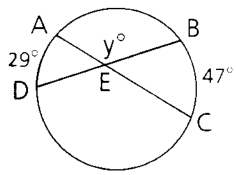
Part Two: Sample Problems

Problem 1 *Given:* $\overline{AB}$ *is a diameter of* $\odot$P.
$\overset{\frown}{BD} = 20°$, $\overset{\frown}{DE} = 104°$
Find: $m\angle C$



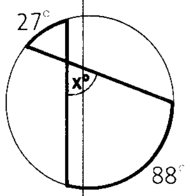
Solution First find $m\overset{\frown}{EA}$.
$m\overset{\frown}{AEB} = 180$, so $m\overset{\frown}{EA} = 180 - (104 + 20) = 56$.
Thus, $m\angle C = \frac{1}{2}(m\overset{\frown}{EA} - m\overset{\frown}{DB}) = \frac{1}{2}(56 - 20) = 18$.

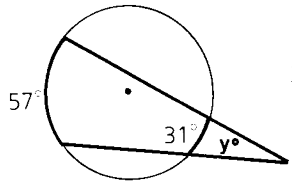
Problem 2 *Find y.*

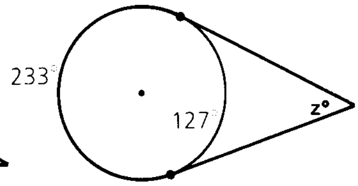


Solution Find $m\angle BEC$ first.
$m\angle BEC = \frac{1}{2}(29 + 47) = 38$
Thus, $y = 180 - m\angle BEC = 142$.

Problem 3 **a** *Find x.* **b** *Find y.* **c** *Find z.*







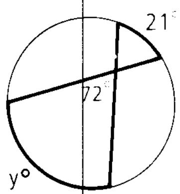
Solution

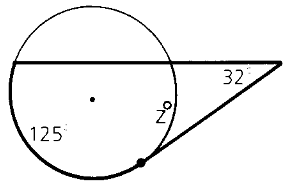
a $x = \frac{1}{2}(88 + 27)$
$= 57\frac{1}{2}$

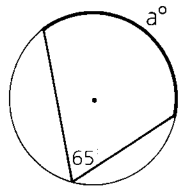
b $y = \frac{1}{2}(57 - 31)$
$= 13$

c $z = \frac{1}{2}(233 - 127)$
$= 53$

Problem 4 **a** *Find y.* **b** *Find z.* **c** *Find a.*







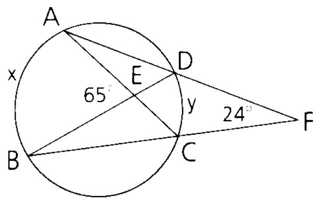
Solution

a $\frac{1}{2}(21 + y) = 72$
$21 + y = 144$
$y = 123$

b $\frac{1}{2}(125 - z) = 32$
$125 - z = 64$
$z = 61$

c $\frac{1}{2}a = 65$
$a = 130$

Problem 5 *Find m$\overset{\frown}{AB}$ and m$\overset{\frown}{CD}$.*



Solution Let m$\overset{\frown}{AB}$ = x and m$\overset{\frown}{CD}$ = y.
Then $\frac{1}{2}(x + y) = 65$ and $\frac{1}{2}(x - y) = 24$.
So $x + y = 130$ and $x - y = 48$.

$$\begin{aligned} x + y &= 130 \\ x - y &= 48 \\ \hline 2x &= 178 \quad \text{Add the equations.} \\ x &= 89 \end{aligned}$$

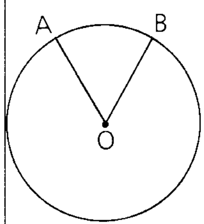
$$\begin{aligned} 89 + y &= 130 \\ y &= 41 \end{aligned}$$

Thus, m$\overset{\frown}{AB}$ = 89 and m$\overset{\frown}{CD}$ = 41.

Part Three: Problem Sets

Problem Set A

1 *Vertex at center:*
Given: $\overset{\frown}{AB} = 62°$
Find: m∠O

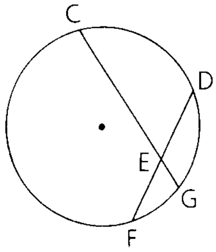


Problem Set A, *continued*

2 *Vertex inside:*

Given: $\overset{\frown}{CD} = 100°$, $\overset{\frown}{FG} = 30°$

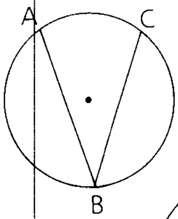
Find: m∠CED



3 *Vertex on:*

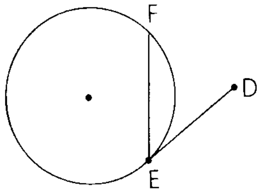
a Given: $\overset{\frown}{AC} = 70°$

Find: m∠B



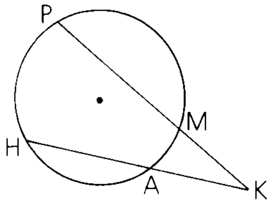
b Given: $\overline{DE}$ is tangent at E.
$\overset{\frown}{EF} = 150°$

Find: m∠DEF



4 *Vertex outside:*

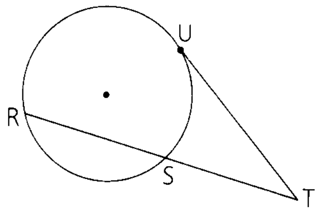
a



Given: $\overset{\frown}{HP} = 120°$,
$\overset{\frown}{AM} = 36°$

Find: m∠K

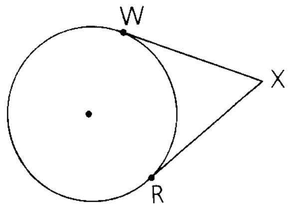
b



Given: $\overline{TU}$ is tangent at U.
$\overset{\frown}{RU} = 160°$,
$\overset{\frown}{SU} = 60°$

Find: m∠T

c

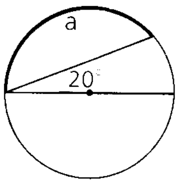


Given: W and R are points of contact.
$\overset{\frown}{WR} = 140°$

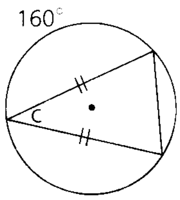
Find: m∠X

5 Find the measure of each angle or arc that is labeled with a letter.

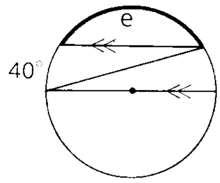
a



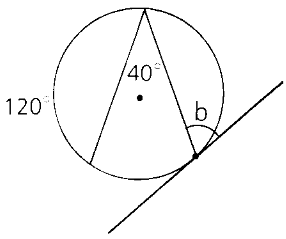
c



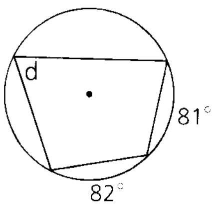
e



b

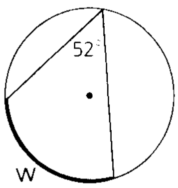


d

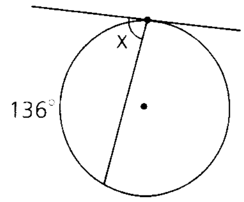


6 Find the measure of each angle or arc that is labeled with a letter.

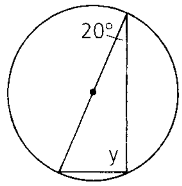
a



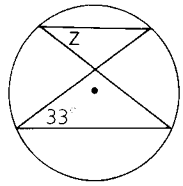
b



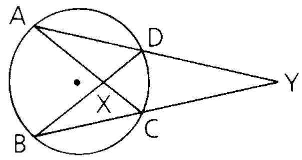
c



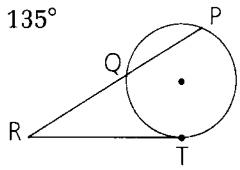
d



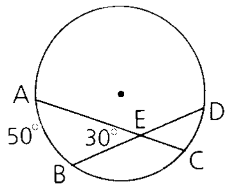
7 Given: $\overset{\frown}{AB} = 108°$, $\overset{\frown}{CD} = 62°$
Find: $\angle AXB$ and $\angle Y$



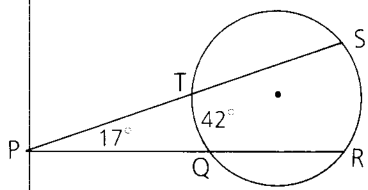
8 Given: $\overset{\frown}{TP} = 170°$, $\overset{\frown}{PQ} = 135°$
Find: $\angle R$



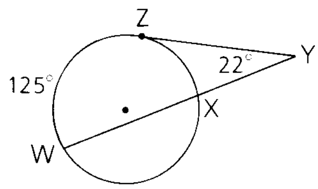
9 Given: $\angle AEB = 30°$,
$\overset{\frown}{AB} = 50°$
Find: $\overset{\frown}{CD}$



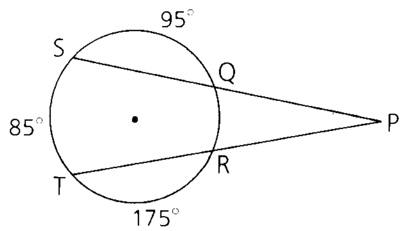
10 Given: $\angle P = 17°$,
$\overset{\frown}{TQ} = 42°$
Find: $\overset{\frown}{SR}$



11 If $\angle Y = 22°$, $\overset{\frown}{WZ} = 125°$, and $\overleftrightarrow{YZ}$ is tangent at Z, find $\overset{\frown}{XZ}$.



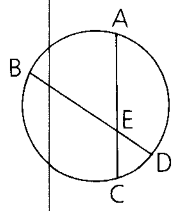
12 If $\overset{\frown}{ST} = 85°$, $\overset{\frown}{SQ} = 95°$, and $\overset{\frown}{TR} = 175°$, find $\angle P$.



Problem Set A, *continued*

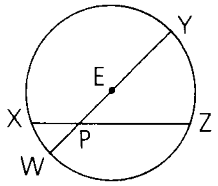
13 Given: $\widehat{AB} = 85°$,
$\widehat{CD} = 25°$

Find: $\angle AED$



14 Given: $\overline{WY}$ is a diameter of $\odot E$.
$\widehat{WX} = 50°$, $\angle XPY = 120°$

Find: $\widehat{WZ}$

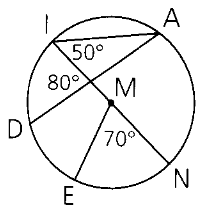


15 A circle is divided into three arcs in the ratio of 3:4:5. A tangent-chord angle intercepts the largest of the three arcs. Find the measure of the tangent-chord angle.

16 An inscribed angle intercepts an arc that is $\frac{1}{9}$ of the circle. Find the measure of the inscribed angle.

17 If a point is chosen at random on $\odot M$, what is the probability that it lies on

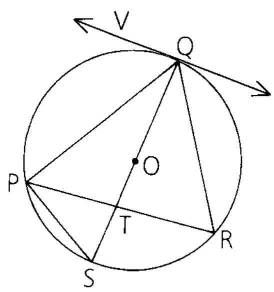
a $\widehat{IAN}$ **b** $\widehat{AN}$ **c** $\widehat{ID}$ **d** $\widehat{IE}$



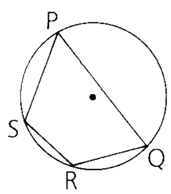
Problem Set B

18 Given: $\overleftrightarrow{VQ}$ is tangent to $\odot O$ at Q.
$\overline{QS}$ is a diameter of $\odot O$.
$\widehat{PQ} = 115°$; $\angle RPS = 36°$

Find: **a** $\angle R$ **e** $\angle QPR$ **i** $\widehat{PRQ}$
b $\angle S$ **f** $\angle QPS$ **j** $\widehat{RSP}$
c $\widehat{SR}$ **g** $\angle QTP$ **k** $\angle VQS$
d $\widehat{QR}$ **h** $\angle PQV$ **l** $\angle QOP$



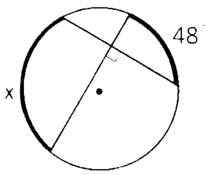
19 Given $m\angle P = 60$ and $m\widehat{PSR} = 128$, find $m\angle Q$, $m\angle R$, and $m\angle S$.



20 The major arc cut off by two tangents to a circle from an outside point is five thirds of the minor arc. Find the angle formed by the tangents.

21 Find the measure of each arc or angle labeled with a letter.

a



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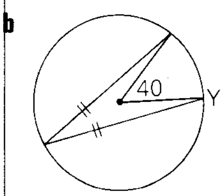
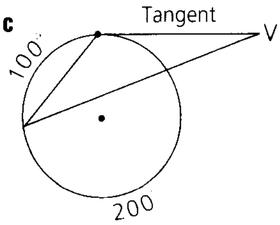


c



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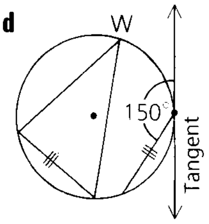
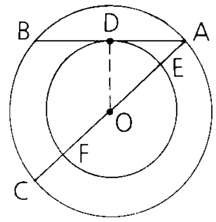
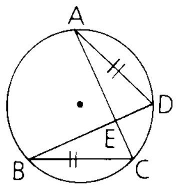


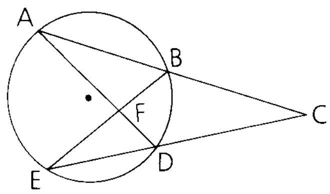
22 Given circles concentric at O, $\overline{AB}$ tangent to the inner circle, and $\widehat{BC} = 84°$, find the measures of $\angle A$, $\widehat{DE}$, and $\widehat{DF}$.



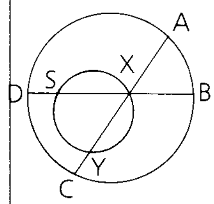
23 Given: $\widehat{AB} = 92°$,
$\angle AEB = 82°$
Find: $\widehat{AD}$



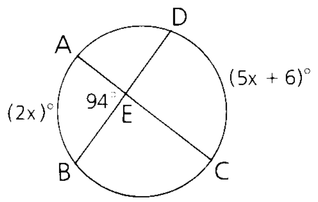
24 Given: $\angle AFE = 89°$,
$\angle C = 15°$
Find: $\widehat{AE}$ and $\widehat{BD}$



25 Given: $\widehat{SY} = 112°$,
$\widehat{DC} = 87°$
Find: $\widehat{AB}$



26 If $\widehat{DC} = (5x + 6)°$, $\widehat{AB} = (2x)°$, and $\angle AEB = 94°$, find $\widehat{AB}$.



27 A secant-secant angle intercepts arcs that are $\frac{3}{5}$ and $\frac{3}{8}$ of the circle. If a chord-chord angle and its vertical angle intercept the same arcs, what is the measure of the chord-chord angle?

28 $\triangle ABC$ is inscribed in a circle (all sides are chords), AB = 12, AC = 6, and BC = $6\sqrt{3}$. Find $m\widehat{BC}$.

Problem Set B, *continued*

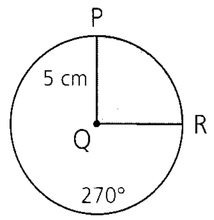
29 **a** An angle is inscribed in a circle and intercepts an arc of 140°. Find the measure of the angle.

b An angle is inscribed in a 140° arc (the vertex is on the arc and the sides contain the endpoints of the arc). Find the measure of the angle.

30 **a** Find the area and the circumference of ⊙Q to the nearest tenth.

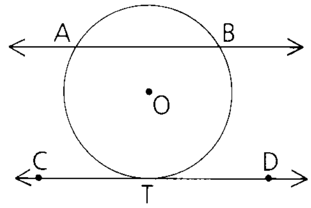
b Find the area of the shaded region to the nearest tenth.

c Find the length of $\widehat{PR}$ to the nearest tenth.



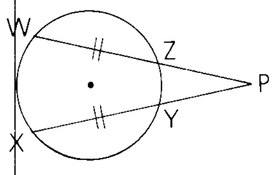
Problem Set C

31 Given: $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$;
$\overleftrightarrow{DC}$ is tangent to ⊙O at T.
Conclusion: $\widehat{AT} \cong \widehat{BT}$



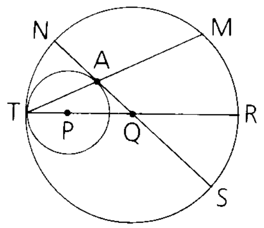
32 A quadrilateral ABCD is inscribed in a circle. Its diagonals intersect at X. If $\widehat{AB} = 100°$, $\widehat{BC} = 50°$, and $\overline{AD} \cong \overline{BD}$, find m∠DXC.

33 Given: $\overline{WZ} \cong \overline{XY}$,
$\widehat{WXY} = 200°$
Find: ∠P

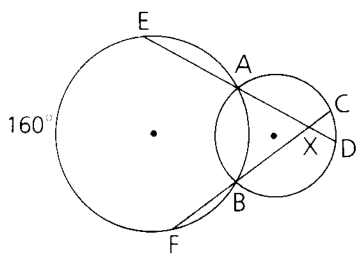


34 A secant and a tangent to a circle intersect to form an angle of 38°. If the measures of the arcs intercepted by this angle are in a ratio of 2:1, find the measure of the third arc.

35 Given: ⓢ P and Q are internally tangent at T.
Diameter $\overline{NS}$ of ⊙Q is tangent to ⊙P at A.
m$\widehat{MR}$ = 42; $\overline{TM}$ passes through A.
Find: m$\widehat{NM}$



36 The two circles shown intersect at A and B. If ∠AXB = 70°, $\widehat{CD} = 20°$, and $\widehat{EF} = 160°$, find the difference between the measures of $\widehat{AB}$ of the smaller circle and $\widehat{AB}$ of the larger circle.



10.6 More Angle-Arc Theorems

Objectives

After studying this section, you will be able to

- Recognize congruent inscribed and tangent-chord angles
- Determine the measure of an angle inscribed in a semicircle
- Apply the relationship between the measures of a tangent-tangent angle and its minor arc

Part One: Introduction

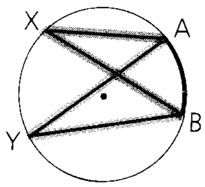
Congruent Inscribed and Tangent-Chord Angles

Our knowledge of the relationships between angles and their intercepted arcs leads easily to the next two theorems.

Theorem 89 ***If two inscribed or tangent-chord angles intercept the same arc, then they are congruent.***

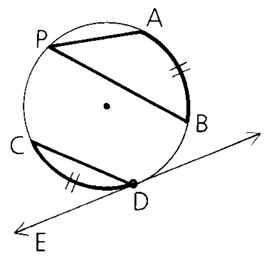
Given: X and Y are inscribed angles intercepting arc AB.

Conclusion: $\angle X \cong \angle Y$



Theorem 90 ***If two inscribed or tangent-chord angles intercept congruent arcs, then they are congruent.***

If $\overleftrightarrow{ED}$ is the tangent at D and $\widehat{AB} \cong \widehat{CD}$, we may conclude that $\angle P \cong \angle CDE$.

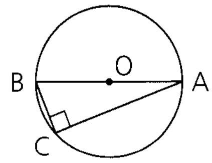


Angles Inscribed in Semicircles

All angles inscribed in semicircles have the same measure. What do you think that measure might be?

Theorem 91 ***An angle inscribed in a semicircle is a right angle.***

Given: $\overline{AB}$ is a diameter of ⊙O.
Prove: ∠C is a right angle.

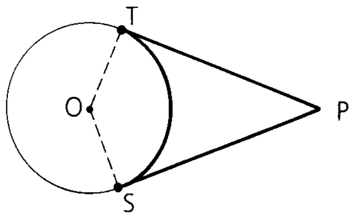


A Special Theorem About Tangent-Tangent Angles

A tangent-tangent angle has a special relationship with its minor arc.

Theorem 92 ***The sum of the measures of a tangent-tangent angle and its minor arc is 180.***

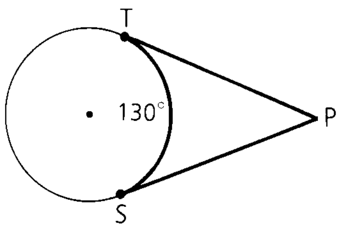
Given: $\overline{PT}$ and $\overline{PS}$ are tangent to circle O.
Prove: $m\angle P + m\overset{\frown}{TS} = 180$



Proof: Since the sum of the measures of the angles in quadrilateral SOTP is 360 and since ∠T and ∠S are right angles, $m\angle P + m\angle O = 180$. Therefore, $m\angle P + m\overset{\frown}{TS} = 180$.

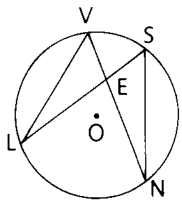
Example $\overleftrightarrow{PT}$ *and* $\overleftrightarrow{PS}$ *are tangents at* T *and* S. *Find* m∠P.

$$m\angle P + m\overset{\frown}{TS} = 180$$
$$m\angle P + 130 = 180$$
$$m\angle P = 50$$



Part Two: Sample Problems

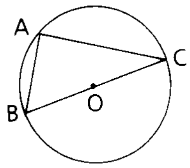
Problem 1 *Given:* ⊙O
Conclusion: △LVE ~ △NSE,
EV · EN = EL · SE



Proof

1 ⊙O	1 Given
2 ∠V ≅ ∠S	2 If two inscribed ∠s intercept the same arc, they are ≅.
3 ∠L ≅ ∠N	3 Same as 2
4 △LVE ~ △NSE	4 AA (2, 3)
5 $\frac{EV}{SE} = \frac{EL}{EN}$	5 Ratios of corresponding sides of ~ ▲s are =.
6 EV · EN = EL · SE	6 Means-Extremes Products Theorem

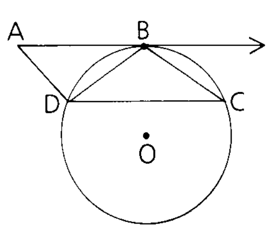
Problem 2 *In circle O, $\overline{BC}$ is a diameter and the radius of the circle is 20.5 mm. Chord $\overline{AC}$ has a length of 40 mm. Find AB.*



Solution Since $\angle A$ is inscribed in a semicircle, it is a right angle. By the Pythagorean Theorem,

$$(AB)^2 + (AC)^2 = (BC)^2$$
$$(AB)^2 + 40^2 = 41^2$$
$$AB = 9 \text{ mm}$$

Problem 3 *Given: $\odot$O with $\overleftrightarrow{AB}$ tangent at B, $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$*
Prove: $\angle C \cong \angle BDC$



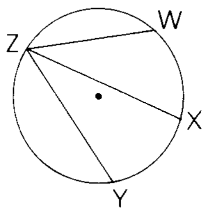
Proof

1 $\overleftrightarrow{AB}$ is tangent to $\odot$O.	1 Given
2 $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$	2 Given
3 $\angle ABD \cong \angle BDC$	3 $\parallel$ lines $\Rightarrow$ alt. int. $\angle$s $\cong$
4 $\angle C \cong \angle ABD$	4 If an inscribed $\angle$ and a tangent-chord $\angle$ intercept the same arc, they are $\cong$.
5 $\angle C \cong \angle BDC$	5 Transitive Property

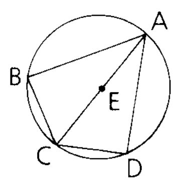
Part Three: Problem Sets

Problem Set A

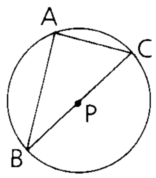
1 Given: X is the midpt. of $\widehat{WY}$.
Prove: $\overrightarrow{ZX}$ bisects $\angle WZY$.



2 Given: $\odot$E with diameter $\overline{AC}$, $\overline{BC} \cong \overline{CD}$
Conclusion: $\triangle ABC \cong \triangle ADC$



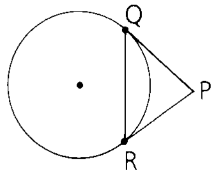
3 In $\odot$P, $\overline{BC}$ is a diameter, AC = 12 mm, and BA = 16 mm. Find the radius of the circle.



Problem Set A, *continued*

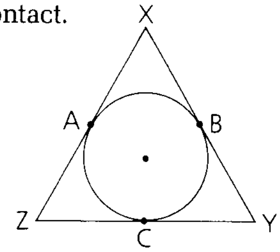
4 Given: $\overline{PQ}$ and $\overline{PR}$ are tangent segments.
$\overset{\frown}{QR} = 163°$

Find: **a** $\angle P$
b $\angle PQR$



5 Given: A, B, and C are points of contact.
$\overset{\frown}{AB} = 145°$, $\angle Y = 48°$

Find: $\angle Z$

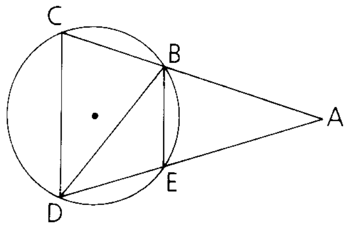


6 Given: $\overset{\frown}{BC} \cong \overset{\frown}{ED}$, $AB = 8$,
$BC = 4$, $CD = 9$

a Are $\overline{BE}$ and $\overline{CD}$ parallel?

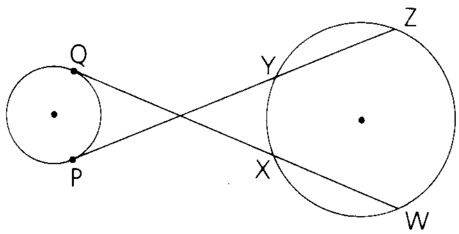
b Find BE.

c Is $\triangle ACD$ scalene?

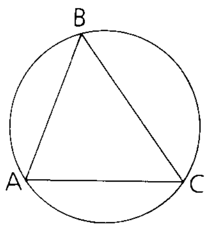


7 Given: $\overleftrightarrow{PY}$ and $\overleftrightarrow{QW}$ are tangents.
$\overset{\frown}{WZ} = 126°$, $\overset{\frown}{XY} = 40°$

Find: $\overset{\frown}{PQ}$



8 If $\triangle ABC$ is inscribed in a circle and $\overset{\frown}{AC} \cong \overset{\frown}{AB}$, tell whether each of the following must be true, could be true, or cannot be true.



a $\overline{AB} \cong \overline{AC}$

b $\overline{AC} \cong \overline{BC}$

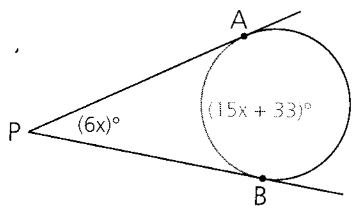
c $\overline{AB}$ and $\overline{AC}$ are equidistant from the center of the circle.

d $\angle B \cong \angle C$

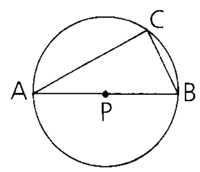
e $\angle BAC$ is a right angle.

f $\angle ABC$ is a right angle.

9 In the figure shown, find $m\angle P$.



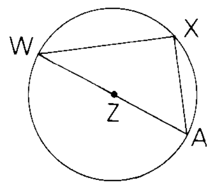
10 If $\overline{AB}$ is a diameter of ⊙P, CB = 1.5 m, and CA = 2 m, find the radius of ⊙P.



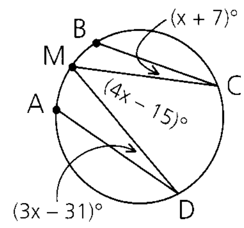
11 The radius of ⊙Z is 6 cm and $\overset{\frown}{WX} = 120°$.

Find: **a** AX

b The perimeter of △WAX



12 M is the midpoint of $\overset{\frown}{AB}$. Find $m\overset{\frown}{CD}$.



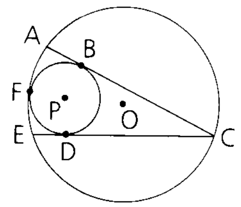
Problem Set B

13 A rectangle with dimensions 18 by 24 is inscribed in a circle. Find the radius of the circle.

14 A square is inscribed in a circle with a radius of 10. Find the length of a side of the square.

15 Quadrilateral ABCD is inscribed in circle O. AB = 12, BC = 16, CD = 10, and ∠ABC is a right angle. Find the measure of $\overline{AD}$ in simplified radical form.

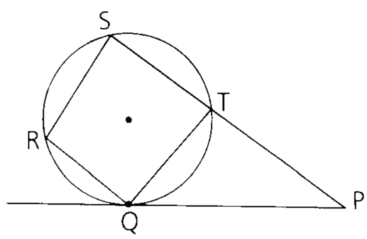
16 Circles O and P are tangent at F. $\overline{AC}$ and $\overline{CE}$ are tangent to ⊙P at B and D. If $\overset{\frown}{DFB} = 223°$, find $\overset{\frown}{AE}$.



17 Given: ∠S = 88°, $\overset{\frown}{QT} = 104°$, $\overset{\frown}{ST} = 94°$, tangent $\overline{PQ}$

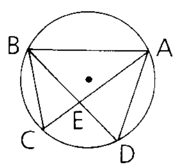
Find: **a** ∠P

b ∠STQ



18 Given: $\overset{\frown}{BC} \cong \overset{\frown}{CD}$

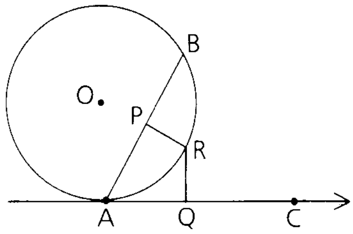
Conclusion: △ABC ~ △AED



Problem Set B, *continued*

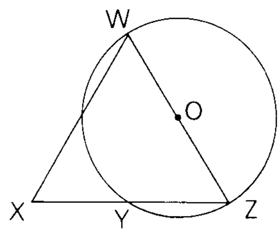
19 Given: $\overleftrightarrow{AC}$ is tangent at A. $\angle APR$ and $\angle AQR$ are right $\angle$s. R is the midpoint of $\overset{\frown}{AB}$.

Conclusion: $\overline{PR} \cong \overline{RQ}$ (Hint: Draw $\overline{AR}$.)



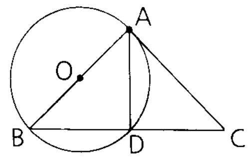
20 Given: $\triangle WXZ$ is isosceles, with $\overline{WX} \cong \overline{WZ}$. $\overline{WZ}$ is a diameter of $\odot O$.

Prove: Y is the midpoint of $\overline{XZ}$. (Hint: Draw $\overline{WY}$.)



21 Given: $\overline{AC}$ is tangent to $\odot O$ at A.

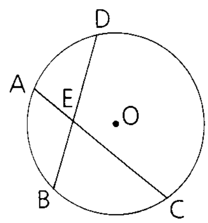
Conclusion: $\triangle ADC \sim \triangle BDA$



Problem Set C

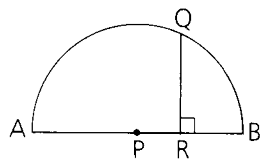
22 Given: $\odot O$, with chords $\overline{AC}$ and $\overline{BD}$ intersecting at E

Prove: **a** $m\overset{\frown}{AB} + m\overset{\frown}{CD} = 2(m\angle CED)$
b $AE \cdot EC = BE \cdot ED$

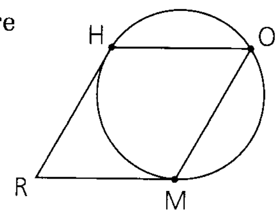


23 Given: $\overline{AB}$ is a diameter of $\odot P$. $QR = 6$, $AB = 13$, $\overline{QR} \perp \overline{AB}$

Find: RB.



24 RHOM is a rhombus. $\overline{RH}$ and $\overline{RM}$ are tangents. Find $m\overset{\frown}{HM}$.

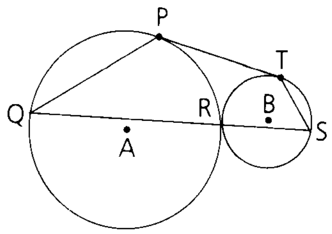


25 Given: $\triangle ABC$ is inscribed in $\odot P$. $\overline{AE}$ and $\overline{CD}$ are chords such that $\overline{AE} \perp \overline{BC}$ and $\overline{CD} \perp \overline{AB}$.

Prove: $\overset{\frown}{BD} \cong \overset{\frown}{BE}$

26 Two circles are internally tangent, and the center of the larger circle is on the smaller circle. Prove that any chord that has one endpoint at the point of tangency is bisected by the smaller circle.

27 Given: ⊙A is tangent to ⊙B at R.
$\overline{PT}$ is a common external tangent at P and T.
$\angle Q = 43°$
Find: $\angle S$



28 Given: $\overline{IT}$ is tangent to the circle.
$\overrightarrow{TS}$ bisects $\angle ATM$.
Prove: $\triangle SIT$ is isosceles.